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Seidel Energy of k-fold and Strong k-fold Graphs

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Abstract

The Seidel energy of a graph is the sum of absolute values of the eigenvalues of its Seidel matrix. In this paper, an explicit expression for the Seidel energy of k-fold graphs and strong k-fold graphs is obtained. As a consequence, certain Seidel equienergetic graphs are characterized. Moreover, some new class of Seidel equienergetic graphs are presented.

Keywords: Seidel energy, Double graph, k-fold graph, Strong double graph, Strong k-fold graph.

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1. Introduction

The most elaborated matrix corresponding to a graph G with n vertices is the *adjacency matrix* $A(G) = [a_{ij}]$, defined by $a_{ij} = 1$ if a vertex v_i is adjacent to a vertex v_j and 0 otherwise. Another well known matrix corresponding to a graph is the *Seidel matrix* $S(G)$ [20] introduced by van Lint and Seidel in 1966. It is defined as $S(G) = J_n - I - 2A(G)$, where J_n is the matrix with all its entries equal to 1 and I is an identity matrix both of same order $n \times n$. The one of important spectral properties of Seidel matrix is that the multiplicity of least Seidel eigenvalue has a connection with equiangular lines in Euclidean space [3]. The *energy* of a graph G is the sum of absolute values of the eigenvalues of G [5], this quantity is defined in connection with molecular chemistry and gained its own importance in the spectral graph theory. Haemers introduced the *Seidel energy* [6] of a graph G, defined as sum of absolute values of the Seidel eigenvalues of G and shown a connection with the energy of G. The study on Seidel energy of a graph can be found in [1, 2, 11, 16, 19] and references therein. In the study of Seidel energy of a graph, finding the class of graphs with different Seidel eigenvalues which have same Seidel energy is an interesting direction. In this paper, the Seidel energy of k-fold graph and strong k-fold graph is presented in terms of Seidel energy of underlying graph together with some other graph parameters. As a result we characterize some class of graphs with same Seidel energy.

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2. Preliminaries

All the graphs in this paper are simple and undirected. Let the $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of a graph G with n vertices $v_1, v_2, \ldots, v_n$. The *degree* d_i of a vertex v_i is the number of edges which are incident with v_i. A graph G is said to be r-*regular* if $d_i = r$ to each vertex $v_i \in V$. The eigenvalues of a graph are the eigenvalues of its adjacency matrix. The *Seidel eigenvalues* of a graph are the eigenvalues of its Seidel matrix and are denoted by $\theta_1, \theta_2, \ldots, \theta_n$. If all the Seidel eigenvalues are integers, then the corresponding graph is called *Seidel integral* graph. The *Seidel energy* of a graph G of order n is defined as $\mathcal{E}_S(G) = \sum_{j=1}^{n} |\theta_j|$. Two graphs G_1 and G_2 with the same number of vertices are said to be Seidel equienergetic if $\mathcal{E}_S(G_1) = \mathcal{E}_S(G_2)$. Let n_S^-, n_S^0 and n_S^+ respectively, denote the number of negative, zero and positive Seidel eigenvalues of G. Let the graphs K_n and K_{n_1,n_2} denote the complete graph with n vertices and the complete bipartite graph with $n_1 + n_2$ vertices respectively. For other notation, terminology and the results related to the spectra of graphs, we follow [4].

Definition 2.1. [7] The *line graph* $\mathcal{L}(G)$ of a graph G is the graph with vertex set same as the edge set of G in which two vertices are adjacent if and only if the corresponding edges in G have a vertex in common. The k-*th iterated line graph* of G for $k = 0, 1, 2, \ldots$ is defined as $\mathcal{L}^k(G) \equiv \mathcal{L}(\mathcal{L}^{k-1}(G))$, where $\mathcal{L}^0(G) \equiv G$ and $\mathcal{L}^1(G) \equiv \mathcal{L}(G)$.

Definition 2.2. [9] Let the vertex set of a graph G be $V(G) = \{v_1, v_2, \ldots, v_n\}$. For $k \geqslant 2$, the k-*fold graph* $D_k[G]$ of a graph G is obtained by taking k copies of G in which a vertex v_i in one copy is adjacent to a vertex v_j in other copies if and only if v_i is adjacent to v_j in G.

It is noted that the adjacency matrix of $D_k[G]$ is $A(D_k[G])$=$J_k \otimes A(G)$, where $\otimes$ denotes the Kronecker product. If $k = 2$, we get the *double graph* $D(G)$ [10], that is, $D_2[G] \equiv D(G)$.

Definition 2.3. Let the vertex set of a graph G be $V(G) = \{v_1, v_2, \ldots, v_n\}$. For $k \geqslant 2$, the *strong* k-*fold graph* $Sd_k[G]$ of a graph G is obtained by taking k copies of G in which a vertex v_i in one copy is adjacent to a vertex v_j in other copies if and only if v_i is adjacent to v_j in G or $i = j$.

It is noted that the adjacency matrix of $Sd_k[G]$ is $A(Sd_k[G])$=$J_k \otimes (A(G) + I) - I \otimes I$. If $k = 2$, we get the *strong double graph* $Sd(G)$ [10, 12], that is, $Sd_2[G] \equiv Sd(G)$.

Lemma 2.4. [19] *Let the Seidel eigenvalues of a graph* G *with* n *vertices be* θ_j, $1 \leqslant j \leqslant n$. *Then for* $k \geqslant 2$, *the Seidel eigenvalues of* $D_k[G]$ *are* $k\theta_j + (k-1)$, $1 \leqslant j \leqslant n$ *and* -1 $(nk - n$ *times*$)$.

Lemma 2.5. [19] *Let the Seidel eigenvalues of a graph* G *with* n *vertices be* θ_j, $1 \leqslant j \leqslant n$. *Then for* $k \geqslant 2$, *the Seidel eigenvalues of* $Sd_k[G]$ *are* $k\theta_j - (k-1)$, $1 \leqslant j \leqslant n$ *and* 1 $(nk - n$ *times*$)$.

Theorem 2.6. [3] *Let the eigenvalues of an* r*-regular graph* G *with* n *vertices be* r *and* λ_i, $2 \leqslant i \leqslant n$. *Then the Seidel eigenvalues of* G *are* $n - 2r - 1$ *and* $-1 - 2\lambda_i$, $2 \leqslant i \leqslant n$.

Theorem 2.7. [15] *Let* G *be a graph with* n_0 *number of vertices and* m_0 *number of edges such that* $d_i + d_j \geqslant 6$ *to each edge* $e = v_i v_j$ *in* G. *Then the iterated line graphs* $\mathcal{L}^k(G)$ *have all the negative eigenvalues equal to* -2 *with the multiplicity* $m_{k-1} - n_{k-1}$ *for all* $k \geqslant 2$, *where* n_k *and* m_k *denote the number of vertices and the number of edges of* $\mathcal{L}^k(G)$ *respectively.*

Theorem 2.8. [16] *Let the graphs* G_1 *and* G_2 *be* r*-regular with the same number of vertices* n *and* $r \geqslant 3$. *Then* $\mathcal{E}_S(\mathcal{L}^k(G_1)) = \mathcal{E}_S(\mathcal{L}^k(G_2))$ *for all* $k \geqslant 2$.

3. Main Results

In the following, we give an explicit expression for Seidel energy of k-fold graph $D_k[G]$ in terms of Seidel energy of G for any graph G.
Let $n_\theta(\mathbf{I})$ denotes the number of Seidel eigenvalues of G which belongs to the interval $\mathbf{I}$ and let $\nu = 1 - \frac{1}{k}$, $k \geqslant 2$.

Theorem 3.1. *Let the Seidel eigenvalues of* G *be* θ_j, $1 \leqslant j \leqslant n$. *Then for* $k \geqslant 2$,

$$\mathcal{E}_S(D_k[G]) = k\left(2n\nu + \mathcal{E}_S(G) - 2\nu n_S^- + 2\sum_{\theta_j \in (-\nu,0)} (\theta_j + \nu)\right).$$

Proof. Let $\theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_n$ be the Seidel eigenvalues of G. By definition of Seidel energy of a graph, we have

$$\begin{aligned}
\mathcal{E}_S(D_k[G]) &= nk - n + \sum_{j=1}^{n} |k\theta_j + (k-1)| \quad \text{by Lemma 2.4} \\
&= kn\nu + k\sum_{j=1}^{n} |\nu + \theta_j| \\
&= k\left(n\nu + \sum_{\theta_j \leqslant -\nu} (-\nu - \theta_j) + \sum_{\theta_j > -\nu} (\nu + \theta_j)\right) \\
&= k\left(n\nu - \nu n_\theta([\theta_n, -\nu]) + \sum_{\theta_j \leqslant -\nu} |\theta_j| + \nu n_\theta((-\nu, \theta_1]) + \sum_{\theta_j \in (-\nu,0)} \theta_j + \sum_{\theta_j \geqslant 0} \theta_j\right),
\end{aligned}$$

where $n_\theta([\theta_n, -\nu]) = 0$ if $\theta_n \geqslant -\nu$. The Seidel energy of a graph G can be expressed as

$$\mathcal{E}_S(G) = \sum_{j=1}^{n} |\theta_j| = \sum_{\theta_j \leqslant -\nu} |\theta_j| + \sum_{\theta_j \in (-\nu,0)} |\theta_j| + \sum_{\theta_j \geqslant 0} \theta_j, \text{ with this we get}$$

$$\begin{aligned}
\mathcal{E}_S(D_k[G]) &= k\left(n\nu - \nu n_\theta([\theta_n, -\nu]) + \nu n_\theta((-\nu, \theta_1]) + \sum_{\theta_j \in (-\nu,0)} \theta_j + \mathcal{E}_S(G) - \sum_{\theta_j \in (-\nu,0)} |\theta_j|\right) \\
&= k\left(n\nu - \nu n_\theta([\theta_n, -\nu]) + \nu n - \nu n_\theta([\theta_n, -\nu]) + 2\sum_{\theta_j \in (-\nu,0)} \theta_j + \mathcal{E}_S(G)\right) \\
&= k\left(2n\nu + \mathcal{E}_S(G) - 2\Big(\nu n_\theta([\theta_n, -\nu]) - \sum_{\theta_j \in (-\nu,0)} \theta_j\Big)\right). \qquad (3.1)
\end{aligned}$$

The total number of Seidel eigenvalues n of a graph G can be expressed as

$$\begin{aligned}
n &= n_\theta([\theta_n, -\nu]) + n_\theta((-\nu, 0)) + n_S^0 + n_S^+ \quad \text{or} \\
n_\theta([\theta_n, -\nu]) &= n - n_S^+ - n_S^0 - n_\theta(-\nu, 0) = n_S^- - n_\theta((-\nu, 0)). \qquad (3.2)
\end{aligned}$$

Also, we have

$$\sum_{\theta_j \in (-\nu,0)} (\theta_j + \nu) = \sum_{\theta_j \in (-\nu,0)} \theta_j + \nu n_\theta((-\nu, 0)). \qquad (3.3)$$

Using (3.2) and (3.3) in (3.1), we get

$$\mathcal{E}_S(D_k[G]) = k\left(2n\nu + \mathcal{E}_S(G) - 2\nu n_S^- + 2\sum_{\theta_j \in (-\nu,0)} (\theta_j + \nu)\right)$$

which completes the proof. □

It is easy to observe that to each negative Seidel eigenvalue $\theta_j \in (-\nu, 0)$ we have $0 < \theta_j + \nu < \nu$, which gives $\nu n_S^- > \sum_{\theta_j \in (-\nu,0)} (\theta_j + \nu) > 0$ for any graph G. Using this fact we get the following.

Corollary 3.2. *Let* G *be a graph with* n *vertices. Then for* $k \geqslant 2$*,*

$$2n(k-1) + k\mathcal{E}_S(G) - 2n_S^-(k-1) \leqslant \mathcal{E}_S(D_k[G]) < 2n(k-1) + k\mathcal{E}_S(G).$$

□

It is noted that $(-\nu, 0) \subseteq (-1, 0)$ for $k \geqslant 2$. There are many graphs with no Seidel eigenvalues in the interval $(-1, 0)$, for instance, all Seidel integral graphs. If a graph G has no Seidel eigenvalue in the interval $(-\nu, 0)$ then we have the following.

Corollary 3.3. *Let* G *be a graph with* n *vertices. Then for* $k \geqslant 2$*,* G *has no Seidel eigenvalue in the interval* $(-\nu, 0)$ *if and only if*

$$\mathcal{E}_S(D_k[G]) = 2(k-1)(n - n_S^-) + k\mathcal{E}_S(G).$$

Proof. Proof follows directly from the fact that $\sum_{\theta \in (-\nu,0)} (\theta + \nu) = 0$ if and only if G has no Seidel eigenvalue θ in the interval $(-\nu, 0)$ in the Theorem 3.1. □

It is easy to construct Seidel equienergetic graphs by using Theorem 3.1 with the help of Seidel equienergetic graphs with no Seidel eigenvalues in the interval $(-\nu, 0)$ and having the same number of negative Seidel eigenvalues.
Let the Seidel eigenvalues of two graphs G_1 and G_2 be $\theta_1', \theta_2', \ldots, \theta_n'$ and $\theta_1'', \theta_2'', \ldots, \theta_n''$ and let the number of negative Seidel eigenvalues of G_1 and G_2 be n_{S1}^- and n_{S2}^- respectively.

Corollary 3.4. *Let* G_1 *and* G_2 *be Seidel equienergetic graphs with* n *vertices. Then for* $k \geqslant 2$*, the graphs* $D_k[G_1]$ *and* $D_k[G_2]$ *are Seidel equienergetic if and only if* $\nu n_{S1}^- - \sum_{\theta_j' \in (-\nu,0)} (\theta_j' + \nu) = \nu n_{S2}^- - \sum_{\theta_j'' \in (-\nu,0)} (\theta_j'' + \nu)$*. In particular, if* G_1 *and* G_2 *have no Seidel eigenvalues in the interval* $(-\nu, 0)$ *then for* $k \geqslant 2$*, the graphs* $D_k[G_1]$ *and* $D_k[G_2]$ *are Seidel equienergetic if and only if* $n_{S1}^- = n_{S2}^-$*.* □

Example 3.5. *The graphs* $\mathcal{L}^p(K_{n,n} \square K_{n-1})$ *and* $\mathcal{L}^p(K_{n-1,n-1} \square K_n)$ *are integral Seidel equienergetic graphs with the same number of negative Seidel eigenvalues for all* $n \geqslant 5, p \geqslant 0$ *[13], where* $\square$ *denotes the Cartesian product. Therefore by Corollary 3.4, the graphs* $D_k[\mathcal{L}^p(K_{n,n} \square K_{n-1})]$ *and* $D_k[\mathcal{L}^p(K_{n-1,n-1} \square K_n)]$ *are Seidel equienergetic for all* $k \geqslant 2, n \geqslant 5$ *and* $p \geqslant 0$*.*

There are many non-isomorphic regular graphs with same number of vertices and same degree, see [8, 13, 14, 17, 18]. Ramane et al. in [16] shown a way to construct a large pairs of Seidel equienergetic iterated line graphs by using such regular graphs. In the following, we present another large class of Seidel equienergetic graphs.

Theorem 3.6. *Let the graphs* G_1 *and* G_2 *be two* r*-regular Seidel equienergetic graphs with same number of vertices* n *and* $r \geqslant 3$*. Then the graphs* $D_k[\mathcal{L}^p(G_1)]$ *and* $D_k[\mathcal{L}^p(G_2)]$ *are Seidel equienergetic for all* $k \geqslant 2$ *and* $p \geqslant 2$*.*

Proof. If $r \geqslant 3$ for an r-regular graph G, then the iterated line graphs $\mathcal{L}^p(G)$ are also regular. By Theorem 2.7, the graphs $\mathcal{L}^p(G)$, $p \geqslant 2$ have all negative eigenvalues equal to -2. Now using the Theorem 2.6, it is evident that all the negative Seidel eigenvalues of $\mathcal{L}^p(G)$, $p \geqslant 2$ are less than or equal to -1. Therefore, if the graphs G_1 and G_2 are two r-regular graphs with same number of vertices n and $r \geqslant 3$ then the graphs $\mathcal{L}^p(G_1)$ and $\mathcal{L}^p(G_2)$ have no Seidel eigenvalues in the interval $(-1, 0)$ to each $p \geqslant 2$. Also the graphs $\mathcal{L}^p(G_1)$ and $\mathcal{L}^p(G_2)$ are Seidel equienergetic by Theorem 2.8. Hence by Corollary 3.4 the graphs $D_k[\mathcal{L}^p(G_1)]$ and $D_k[\mathcal{L}^p(G_2)]$ are Seidel equienergetic for all $k \geqslant 2$ and $p \geqslant 2$. □

It is interesting to see the Seidel eigenvalues of $D_k[G]$ of a graph G in the interval $(-1, 0)$.

Proposition 3.7. *If a graph G has no Seidel eigenvalues in the interval $(-1, 0)$, then for $k \geqslant 2$, $D_k[G]$ also have no Seidel eigenvalues in the interval $(-1, 0)$.*

Proof. Proof follows directly from the Seidel eigenvalues of $D_k[G]$ in the Lemma 2.4 if G has no Seidel eigenvalues in the interval $(-1, 0)$. □

In the following, we give an explicit expression for Seidel energy of strong k-fold graph $Sd_k[G]$, $k \geqslant 2$ in terms of Seidel energy of G for any graph G.

Theorem 3.8. *Let the Seidel eigenvalues of G be θ_j, $1 \leqslant j \leqslant n$. If $\theta_j \notin (-\nu, \nu)$ then for $k \geqslant 2$,*

$$\mathcal{E}_S(Sd_k[G]) = 2(k-1)(n - n_S^+) + k\mathcal{E}_S(G).$$

Proof. Let $\theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_n$ be the Seidel eigenvalues of G. If $\theta_j \notin (-\nu, \nu)$, then we have

$$|k\theta_j - (k-1)| = \begin{cases} k|\theta_j| - (k-1) & \text{if } \theta_j \geqslant \nu \\ k|\theta_j| + (k-1) & \text{if } \theta_j \leqslant -\nu \, . \end{cases}$$

By definition of Seidel energy of a graph, we have

$$\begin{aligned}
\mathcal{E}_S(Sd_k[G]) &= nk - n + \sum_{j=1}^{n} |k\theta_j - (k-1)| \quad \text{by Lemma 2.5} \\
&= n(k-1) + \sum_{\theta_j \leqslant -\nu} (k|\theta_j| + (k-1)) + \sum_{\theta_j \geqslant \nu} (k|\theta_j| - (k-1)) \\
&= n(k-1) + k\sum_{\theta_j \leqslant -\nu} |\theta_j| + (k-1)n_\theta([\theta_n, -\nu]) + k\sum_{\theta_j \geqslant \nu} |\theta_j| - (k-1)n_\theta([\nu, \theta_1]) \\
&= n(k-1) + k\mathcal{E}_S(G) + (k-1)(n_\theta([\theta_n, -\nu]) - n_\theta([\nu, \theta_1])).
\end{aligned}$$

If If $\theta_j \notin (-\nu, \nu)$, then total number of Seidel eigenvalues n of a graph G can be expressed as $n = n_\theta([\theta_n, -\nu]) + n_\theta([\nu, \theta_1])$, with this fact we have

$$\begin{aligned}
\mathcal{E}_S(Sd_k[G]) &= n(k-1) + k\mathcal{E}_S(G) + (k-1)(n - n_\theta([\nu, \theta_1]) - n_\theta([\nu, \theta_1])) \\
&= 2n(k-1) + k\mathcal{E}_S(G) - 2(k-1)(n_\theta([\nu, \theta_1])). \\
&= 2n(k-1) + k\mathcal{E}_S(G) - 2(k-1)n_S^+, \quad \text{since } \nu > 0 \text{ and } \theta_j \notin (-\nu, \nu) \\
&= 2(k-1)(n - n_S^+) + k\mathcal{E}_S(G).
\end{aligned}$$

which completes the proof. □

In the following, another class of Seidel equienergetic graphs are characterized. Let the number of positive Seidel eigenvalues of the graphs G_1 and G_2 be n_{S1}^+ and n_{S2}^+ respectively.

Corollary 3.9. *Let G_1 and G_2 be Seidel equienergetic graphs with no Seidel eigenvalues in the interval $(-\nu,\nu)$ and both with n vertices. Then for $k \geqslant 2$, the graphs $Sd_k[G_1]$ and $Sd_k[G_2]$ are Seidel equienergetic if and only if $n^+_{S1} = n^+_{S2}$.* □

Example 3.10. *The graphs $K_{n,n} \boxtimes K_{n-1}$ and $K_{n-1,n-1} \boxtimes K_n$ are integral Seidel equienergetic graphs with the same number of positive Seidel eigenvalues for all $n \geqslant 3$ [13], where $\boxtimes$ denotes the strong product. Therefore by Corollary 3.9, the graphs $Sd_k[K_{n,n} \boxtimes K_{n-1}]$ and $Sd_k[K_{n-1,n-1} \boxtimes K_n]$ are Seidel equienergetic for all $n \geqslant 3$ and $k \geqslant 2$.*

The following is Theorem 2.4 of [19] which is the consequence of Corollary 3.3 and Theorem 3.8.

Theorem 3.11. *Let the Seidel eigenvalues of G be θ_j, $1 \leqslant j \leqslant n$ and $\theta_j \notin (-\nu,\nu)$. Then for $k \geqslant 2$ the graphs $D_k[G]$ and $Sd_k[G]$ are Seidel equienergetic if and only if $n^-_S = n^+_S$.* □

In the following, we present the Seidel energy of $Sd_k[D_k[G]]$, $k \geqslant 2$ for any graph G.

Theorem 3.12. *Let the Seidel eigenvalues of G be θ_j, $1 \leqslant j \leqslant n$. Then for $k \geqslant 2$,*

$$\mathcal{E}_S(Sd_k[D_k[G]]) = 2n(k-1)(2k-1) + k^2\left(\mathcal{E}_S(G) - 2\nu^2 n^-_S + 2\sum_{\theta_j \in (-\nu^2,0)} (\theta_j + \nu^2)\right).$$

Proof. If $\theta_1, \theta_2, \ldots, \theta_n$ are the Seidel eigenvalues of G, then by Lemma 2.4 and Lemma 2.5, the Seidel eigenvalues of $Sd_k[D_k[G]]$ are $k^2\theta_j + (k-1)^2$, $1 \leqslant j \leqslant n$, $1-2k$ ($nk-n$ times) and 1 ($nk^2 - nk$ times) [19]. By definition of Seidel energy of a graph, we have

$$\begin{aligned}\mathcal{E}_S(Sd_k[D_k[G]]) &= nk^2 - nk + (2k-1)(nk-n) + \sum_{j=1}^{n} |k^2\theta_j + (k-1)^2| \\ &= 3nk^2 - 4kn + n + \sum_{j=1}^{n} |k^2\theta_j + (k-1)^2|\end{aligned}$$

Now proceeding similar to that of proof of Theorem 3.1, we get

$$\mathcal{E}_S(Sd_k[D_k[G]]) = 2n(k-1)(2k-1) + k^2\left(\mathcal{E}_S(G) - 2\nu^2 n^-_S + 2\sum_{\theta_j \in (-\nu^2,0)} (\theta_j + \nu^2)\right),$$

which completes the proof. □

Again, it can be seen that to each negative Seidel eigenvalue $\theta_j \in (-\nu^2, 0)$ we have $0 < \theta_j + \nu^2 < \nu^2$, which gives $\nu^2 n^-_S > \sum_{\theta_j \in (-\nu^2,0)} (\theta_j + \nu^2) > 0$ for any graph G. Using this fact we get the following.

Corollary 3.13. *Let G be a graph with n vertices. Then for $k \geqslant 2$,*

$$2n(k-1)(2k-1) + k^2\mathcal{E}_S(G) - 2(k-1)^2 n^-_S \leqslant \mathcal{E}_S(Sd_k[D_k[G]]) < 2n(k-1)(2k-1) + k^2\mathcal{E}_S(G).$$

□

Again, it is noted that $(-\nu^2, 0) \subseteq (-1, 0)$ for $k \geqslant 2$. If a graph G has no Seidel eigenvalue in the interval $(-\nu^2, 0)$ then we have the following.

Corollary 3.14. *Let G be a graph with n vertices. Then for $k \geqslant 2$, G has no Seidel eigenvalue in the interval $(-\nu^2, 0)$ if and only if*

$$\mathcal{E}_S(Sd_k[D_k[G]]) = 2n(k-1)(2k-1) + k^2\mathcal{E}_S(G) - 2(k-1)^2 n^-_S.$$

Proof. Proof follows directly from the fact that $\sum_{\theta \in (-\nu^2, 0)} (\theta + \nu^2) = 0$ if and only if G has no Seidel eigenvalue θ in the interval $(-\nu^2, 0)$ in the Theorem 3.12. □

The following provides a way to construct Seidel equienergetic graphs. Let the Seidel eigenvalues of two graphs G_1 and G_2 be $\theta_1', \theta_2', \ldots, \theta_n'$ and $\theta_1'', \theta_2'', \ldots, \theta_n''$ and let the number of negative Seidel eigenvalues of G_1 and G_2 be n_{S1}^- and n_{S2}^- respectively.

Corollary 3.15. *Let G_1 and G_2 be Seidel equienergetic graphs with n vertices. Then for $k \geqslant 2$, the graphs $Sd_k[D_k[G_1]]$ and $Sd_k[D_k[G_2]]$ are Seidel equienergetic if and only if $\nu^2 n_{S1}^- - \sum_{\theta_j' \in (-\nu^2, 0)} (\theta_j' + \nu^2) = \nu^2 n_{S2}^- - \sum_{\theta_j'' \in (-\nu^2, 0)} (\theta_j'' + \nu^2)$. In particular, if G_1 and G_2 have no Seidel eigenvalues in the interval $(-\nu^2, 0)$ then for $k \geqslant 2$, the graphs $Sd_k[D_k[G_1]]$ and $Sd_k[D_k[G_2]]$ are Seidel equienergetic if and only if $n_{S1}^- = n_{S2}^-$.* □

Example 3.16. *Consider the graphs $\mathcal{L}^p(K_{n,n} \square K_{n-1})$ and $\mathcal{L}^p(K_{n-1,n-1} \square K_n)$ in the example 3.5. By using Corollary 3.15, the graphs $Sd_k[D_k[\mathcal{L}^p(K_{n,n} \square K_{n-1})]]$ and $Sd_k[D_k[\mathcal{L}^p(K_{n-1,n-1} \square K_n)]]$ are Seidel equienergetic for all $k \geqslant 2, n \geqslant 5$ and $p \geqslant 0$.*

In the following, we present another large class of Seidel equienergetic graphs.

Theorem 3.17. *Let the graphs G_1 and G_2 be two r-regular Seidel equienergetic graphs with same number of vertices n and $r \geqslant 3$. Then the graphs $Sd_k[D_k[\mathcal{L}^p(G_1)]]$ and $Sd_k[D_k[\mathcal{L}^p(G_2)]]$ are Seidel equienergetic for all $k \geqslant 2$ and $p \geqslant 2$.*

Proof. Proof follows similar to that of proof of Theorem 3.6 with the help of Corollary 3.15. □

In the following, we present the Seidel energy of $D_k[Sd_k[G]]$, $k \geqslant 2$ for any graph G.

Theorem 3.18. *Let the Seidel eigenvalues of G be θ_j, $1 \leqslant j \leqslant n$. If $\theta_j \notin (-\nu^2, \nu^2)$ then for $k \geqslant 2$,*

$$\mathcal{E}_S(D_k[Sd_k[G]]) = 2n(k-1)(2k-1) + k^2 \mathcal{E}_S(G) - 2(k-1)^2 n_S^+.$$

Proof. If $\theta_1, \theta_2, \ldots, \theta_n$ are the Seidel eigenvalues of G, then by Lemma 2.4 and Lemma 2.5, the Seidel eigenvalues of $D_k[Sd_k[G]]$ are $k^2\theta_j - (k-1)^2$, $1 \leqslant j \leqslant n$, $2k-1$ ($nk - n$ times) and -1 ($nk^2 - nk$ times) [19]. By definition of Seidel energy of a graph, we have

$$\begin{aligned} \mathcal{E}_S(D_k[Sd_k[G]]) &= nk^2 - nk + (2k-1)(nk-n) + \sum_{j=1}^{n} |k^2\theta_j - (k-1)^2| \\ &= 3nk^2 - 4kn + n + \sum_{j=1}^{n} |k^2\theta_j - (k-1)^2| \end{aligned}$$

Now proceeding similar to that of proof of Theorem 3.8, we get

$$\mathcal{E}_S(D_k[Sd_k[G]]) = 2n(k-1)(2k-1) + k^2 \mathcal{E}_S(G) - 2(k-1)^2 n_S^+,$$

which completes the proof. □

In the following, we present another class of Seidel equienergetic graphs. Let the number of positive Seidel eigenvalues of the graphs G_1 and G_2 be n_{S1}^+ and n_{S2}^+ respectively.

Corollary 3.19. *Let G_1 and G_2 be Seidel equienergetic graphs with no Seidel eigenvalues in the interval $(-\nu^2, \nu^2)$ and both with n vertices. Then for $k \geqslant 2$, the graphs $D_k[Sd_k[G_1]]$ and $D_k[Sd_k[G_2]]$ are Seidel equienergetic if and only if $n_{S1}^+ = n_{S2}^+$.* □

Example 3.20. *Consider the graphs $K_{n,n} \boxtimes K_{n-1}$ and $K_{n-1,n-1} \boxtimes K_n$ in the example 3.10. Now by using the Corollary 3.19, the graphs $D_k[Sd_k[K_{n,n} \boxtimes K_{n-1}]]$ and $D_k[Sd_k[K_{n-1,n-1} \boxtimes K_n]]$ are Seidel equienergetic for all $n \geqslant 3$ and $k \geqslant 2$.*

The following is Theorem 2.5 of [19] which is the consequence of Corollary 3.14 and Theorem 3.18.

Theorem 3.21. *Let the Seidel eigenvalues of G be θ_j, $1 \leqslant j \leqslant n$ and $\theta_j \notin (-\nu^2, \nu^2)$. Then for $k \geqslant 2$ the graphs $Sd_k[D_k[G]]$ and $D_k[Sd_k[G]]$ are Seidel equienergetic if and only if $n_S^- = n_S^+$.* □

4. Conclusion

Vaidya and Popat in [19] constructed Seidel equienergetic graphs by using the graphs $D_k[G]$ and $Sd_k[G]$ for any graph G, where $k \geqslant 2$. In this paper, we have given the explicit expressions for the Seidel energy of the graphs $D_k[G]$ and $Sd_k[G]$ and provided a general way to construct certain classes of Seidel equienergetic graphs. As there are many graph operations available in the literature, one can further study the possible relations between the Seidel energy and other various energy types of a graph.

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